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SOME THOUGHTS ON THE THEORY OF COOPERATIVE GAMES

Gerd Jentzsch

Edited by R. J. Aumann

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PREFACE

This is an edited translation of a paper written by Mr. Gerd Jentzsch at the Institute for Applied Economic Research in Tübingen, West Germany, shortly before his death in March, 1959, and issued in report form by that Institute.

Game theory is important in its general applicability to a variety of conflict situations—political and military as well as economic. The present paper is a notable technical contribution in an area of great current interest: conflict situations in which the competing interests are not directly opposed to one another.

Dr. Aumann, who has undertaken the editing of Jentzsch's work, has provided an introduction to this translation, as well as some supplementary notes and many corrections of text. He is a Lecturer in Mathematics at the Hebrew University, Jerusalem, and is now a consultant to The RAND Corporation.

SUMMARY

This work is the result of an attempt to generalize the von Neumann-Morgenstern theory of n-person games by eliminating the requirement that the utility functions be linear, or more generally, by eliminating side payments altogether. It is known that unless suitable restrictions are imposed, the minimax theorem for coalitions will not hold: outcomes will exist which can neither be guaranteed by a coalition nor prevented by the opposing coalition. Jentzsch addresses himself to the task of broadening the class of games considered by von Neumann and Morgenstern while retaining the minimax property for coalitions. His chief result may be described (for games with side payments) by saying that each coalition must have a kind of "social utility function" for money.

CONTENTS

EDITOR'S INTRODUCTION

The author of this paper, Gerd Jentzsch, died while still a young man on March 26, 1959. This is apparently his only publication. Judging from its originality and all-around brilliance, his death was a loss of the first magnitude to game theory.

The von Neumann-Morgenstern (N-M) theory of n-person games [4] is concerned with cooperative games in which side payments are permitted and utility is "unrestrictedly transferable"—that is, each player's utility for money is linear in money.* Jentzsch's investigations grew out of an attempt to generalize the N-M theory either by eliminating the requirement that the utility functions** be linear, or more generally, by eliminating side payments altogether. He notices at the outset that the notion of "effectiveness"—which is crucial in the N-M theory—does not generalize in a straightforward manner. In the classical theory, a coalition K is effective for a payoff vector f if, roughly speaking, the coalition can assure itself of getting at least f. An equivalent definition of effectiveness is that the opposition—the complement of K—cannot prevent K from obtaining at least f. But when utilities are non-linear in money or side payments are forbidden, these two

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*See R. D. Luce and H. Raiffa, Games and Decisions, p. 168.

**By the phrase "utility function" we shall henceforth always mean "utility of money as a function of money."

definitions of effectiveness are in general no longer equivalent—in Jentzsch's terminology, the game need not be "clear" (example 4). Jentzsch addresses himself to the task of broadening the class of games considered by von Neumann and Morgenstern, while still retaining the clearness property.

The chief result is Theorem 21. Rather than stating it here in its most general form, we will describe its application to games with side payments ("money games" for short) in which the utility functions need not be linear. The problem that Jentzsch considers is, what kinds of utility functions of the players will always lead to clear games (as linear utility functions do)? More precisely, what conditions, when placed on the utility functions of the players, will ensure that all money games in which these players participate are clear? The answer is that each coalition must have a kind of "social utility function" for money. For example, this involves the demand that 50 dollars be indifferent—from the point of view of the coalition as a whole—to some probability combination of 0 dollars and 100 dollars (though not necessarily the 1/2—1/2 combination). The sums of money involved (50 dollars, 0 dollars, 100 dollars) are not given to the individual players, but to the coalition as a whole for distribution among its members. "Indifferent" has a very precise meaning here: The two sets of (utility) payoff vectors that can result from the two possibilities must coincide.

The existence of such a "social utility function" is a considerable restriction. Jentzsch remarks without proof that Bernoullian (i.e., logarithmic) individual utility functions lead to a social utility function (examples 11, 16) and that other individual utility functions that lead to a social utility function can be obtained as solutions of a third-order differential equation with one parameter (which he does not specify). These questions must certainly be investigated further. But on the whole, Jentzsch's result shows that clearness is the exception rather than the rule—that games with nonlinear utility functions or without side payments cannot be "expected" to be clear.

The difference between the two kinds of effectiveness was appreciated by others, working independently of Jentzsch, as far back as 1957—which is probably the approximate date of Jentzsch's investigations.* It was explicitly mentioned by Aumann and Peleg [2], who used the names α- and β- effectiveness for the two kinds. A survey of the whole field of cooperative games without side payments is given in [1], which has a bibliography of 15 items; but of this work, Jentzsch knew only of the pioneering investigation of Shapley and Shubik [5]. This is another example of the known phenomenon of the intrinsic

*In fact, the idea is related to Blackwell's approach-ability—excludability theory [3] which appeared already in 1956.

"ripeness" of a scientific concept—leading to simultaneous independent discovery by widely separated investigators. It should be emphasized, though, that it is only in the basic recognition of the difference bewteen α– and β–effectiveness that Jentzsch's work overlaps that of others; the main result of this paper has not been found by anybody else, and appears here for the first time. Indeed other workers have approached the subject from a somewhat different viewpoint—they have tried to "live with" the difference, whereas Jentzsch characterized the conditions under which it could be eliminated (see [1]).

An attempt has been made to keep editorial comment separate from Jentzsch's original text. All the footnotes are the editor's, as are the two "Editor's Notes." The long formal proofs given by Jentzsch for Theorems 10 and 13 have been replaced by short intuitive sketches. There has been some rearranging of the material, and the more straightforward proofs have been left out. Those are all the changes.

Since Jentzsch is interested only in the question of effectiveness, he fixes once for all a coalition K, and considers only the joint strategies of the coalition, the joint strategies of the opposition, and the payoff to the coalition. The resulting formal object is called a "K–game," and this is the object of investigation throughout.*

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*It is formally identical to Blackwell's "game with vector payoffs" [3].

The individual strategies of members of the coalition and of the opposition, and the payoff to the opposition, are of no interest in this context, and are therefore suppressed in the formal model.

R.J.A.

SOME THOUGHTS ON THE THEORY OF COOPERATIVE GAMES

1. K-GAMES

1.1. Definitions

Definition 1. $\Gamma = \Gamma(S,T,r)$ is called a K-game if K is a nonempty finite set, $E = E(K) = \left\{f = (f(k))_{k\epsilon K} \middle| f(k) \text{ real for } k\epsilon K\right\}$, S is a nonempty set, T is a nonempty set, and r is a mapping of SxT into E.

We call K "the coalition", its elements "coalition members", or simply "players." Members of E are called "coalition payoff vectors", or simply "payoff vectors." If the coalition members k anticipate the utilities f(k), then we say that the coalition anticipates the payoff vector f $(f = (f(k))_{k\epsilon K})$. We call S "the set of strategies of the coalition," T "the set of strategies of the opposition," and r "the payoff function." In a K-game the coalition chooses one of its strategies $s\epsilon S$, and the opposition independently chooses one of its strategies $t\epsilon T$. The payoff vector $r(s,t)$ results.

Definition 2. Let Γ and Γ' be two K-games, $\Gamma = \Gamma(S,T,r)$, $\Gamma' = \Gamma(S',T',r')$. We say Γ is "isomorphic" to Γ' if there is a one-to-one mapping σ of S onto S' and a one-to-one mapping τ of T onto T' such that for all $s\epsilon S$, $t\epsilon T$, $s'\epsilon S'$, $t'\epsilon T'$, $r(s,t) = r'(\sigma(s), \tau(t))$, $r'(s',t') = r(\sigma^{-1}(s'), \tau^{-1}(t'))$.

Isomorphic copies of K-games differ only in notation.

Definition 3. Let $f \epsilon E(K)$, $h \epsilon E(K)$. We write

$$f \leq h, \text{ if } f(k) \leq h(k) \text{ for all } k \epsilon K$$
$$f \geq h, \text{ if } f(k) \geq h(k) \text{ for all } k \epsilon K.$$

If the coalition has a choice between payoff vectors f and h, and if $f \leq h$, $f \neq h$, then it will prefer h; h obviously guarantees to each coalition member a utility which is no less, and possibly greater than that received under f. Let us assume that the interests of the opposition are directly opposed to those of the coalition. If the opposition can determine whether the coalition receives f or h, and if $f \leq h$, $f \neq h$, then the opposition will see to it that the coalition only receives f.

Definition 4. Let $\Gamma = \Gamma(S,T,r)$ be a K-game, $S' \subset S$, $T' \subset T$. S' is called "complete" in Γ, if for every $s \epsilon S$ there is an $s' \epsilon S'$ such that $r(s',t) \geq r(s,t)$ for all $t \epsilon T$. T' is called "complete" in Γ, if for every $t \epsilon T$ there is a $t' \epsilon T'$ such that $r(s,t') \leq r(s,t)$ for all $s \epsilon S$.

Definition 5. Let Γ and Γ' be two K-games, $\Gamma = \Gamma(S,T,r)$, $\Gamma' = \Gamma(S',T',r')$. Γ' is called a "deflation" or Γ, if $S' \subset S$, $T' \subset T$, $r'(s,t) = r(s,t)$ for all $s \epsilon S'$, $t \epsilon T'$, and S' and T' are complete in Γ.

The coalition attempts to obtain the best possible payoff vectors and the opposition tries to counteract this. Nothing essential is altered in the possibilities open to either one, if they restrict their strategies to choices from complete subsets of their strategy sets. They can thus change from a K-game to one of its deflations.

Definition 6. Two K-games Γ and Γ' are called "equivalent" if there is a natural number N and a chain of K-games $\Gamma = \Gamma_0, \ldots \Gamma_N = \Gamma'$ such that for all $n = 1, \ldots, N$,

Γ_n is a deflation of Γ_{n-1} ,

Γ_n is an isomorph of Γ_{n-1}, or

Γ_{n-1} is a deflation of Γ_n .

Definition 7. Every K-game $\Gamma(S,T,r)$ has "characteristic sets" $V(s)$, V, U, $U(t)$, where $s \in S$, $t \in T$, defined by

$$V(s) = \left\{f \in E(K) \mid \text{for all } t \in T \quad f \leq r(s,t)\right\},$$

$$U(t) = \left\{f \in E(K) \mid \text{there is an } s \in S \text{ such that } f \leq r(s,t)\right\},$$

$$V = \bigcup_{s \in S} V(s), \qquad U = \bigcap_{t \in T} U(t).$$

These sets may be interpreted in the following manner: $V(s)$ is the set of payoff vectors which the coalition can enforce by using strategy s. V is the set of payoff vectors which the coalition can enforce.

$U(t)$ is the set of payoff vectors which the opposition cannot prevent by using strategy t (for if the coalition discovers the "correct" strategy s, then $f \leq r(s,t)$).
U is the set of payoff vectors which the opposition cannot prevent.

Theorem 1. If U and V are characteristic sets of a K-game (according to Definition 7), then $V \subset U$.

The opposition cannot prevent any payoff vector that the coalition can enforce.

Proof. If $f \in V$, then there is an $s \in S$ such that $f \in V(s)$. Then for every t, $f \leq r(s,t)$. Therefore $f \in U(t)$ for all $t \in T$. So $f \in U$. f was arbitrary, so $V \subset U$.

Definition 8. Let Γ be a K-game, U and V characteristic sets of Γ. Γ is called "clear" if $V = U$.

In a clear K-game the coalition can enforce precisely those payoff vectors that the opposition cannot prevent.

Definition 9. Let $\Gamma(S,T,r)$ be a K-game with the characteristic sets V, U, $U(t^0)$, where $t^0 \in T$. t^0 is called "optimal" and $\Gamma(S,T,r)$ "classical" if $V = U(t^0)$.

By using an optimal strategy the opposition can prevent every payoff vector that the coalition cannot enforce. In a classical K-game the opposition has an optimal strategy.

Theorem 2. Every classical K-game is clear.

Proof. According to Theorem 1, $V \subset U$. According to Definition 7, $U(t^0) \supset U$. From $V = U(t^0)$ it therefore follows that $V = U$.

Theorem 3. The properties "clear" and "classical" are invariant under equivalence (Definition 6).

The proof is straightforward, and is omitted.

The existence of nonclear K-games is shown by the following examples.

Example 1. Let

$$K = \{1\},\ S = \{0,1\},\ T = \{0,1\}\ ,$$

$$r(0,0) = r(1,1) = (1),\quad r(0,1) = r(1,0) = (-1),$$

$$\Gamma_1 = \Gamma(S,T,r).$$

The characteristic sets of Γ_1 are

$$V(0) = V(1) = V = \left\{f \in E(K) \mid f(1) \leq -1\right\},$$

$$U(0) = U(1) = U = \left\{f \in E(K) \mid f(1) \leq +1\right\}\ .$$

Γ_1 is not clear.

Player 1 (as a one-man coalition) plays Matching Pennies against the opposition. Since only pure strategies are permitted, he cannot prevent the loss of his penny; i.e., he cannot "guarantee" more than the loss of the penny.

On the other hand, the opposition cannot prevent player 1 from winning the penny.

<u>Example 2.</u> (See Fig. 1). Let

$K = \{1,2\}$, $S = \{0,1\}$, $T = \{0,1\}$,

$r(0,0) = (2, -1)$, $r(0,1) = (-2,1)$

$r(1,0) = (1, -2)$ $r(1,1) = (-1,2)$

$\Gamma_2 = \Gamma(S.T.r)$.

The characteristic sets of Γ_2 are

$$V(0) = \{f \mid f \leq (-2,-1)\} ,$$

$$V(1) = \{f \mid f \leq (-1,-2)\} ,$$

$$V = V(0) \cup V(1),$$

$$U(0) = \{f \mid f \leq (2,-1)\} ,$$

$$U(1) = \{f \mid f \leq (-1,2)\} ,$$

$$U = \{f \mid f \leq (-1,-1)\} .$$

There is no answer to the question of how a player (or a coalition) should act in a nonclear situation such as that in Matching Pennies. J. von Neumann circumvented these difficulties in the case of a two-person zero sum game by introducing mixed strategies. We will now see if something similar is also possible for K-games.

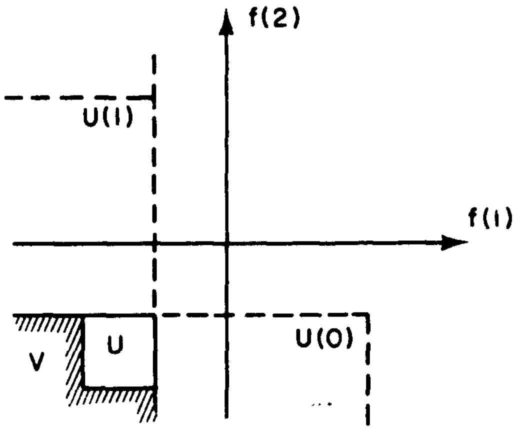

Fig. 1

1.2. Convex Cooperative Games

Definition 10. Addition and scalar multiplication on $E = E(K)$ are defined as is usual for vectors, i.e., $(f + h)(k) = f(k) + h(k)$ and $(cf)(k) = cf(k)$.

Definition 11. Let $\Gamma = \Gamma(S,T,r)$, $\Gamma' = \Gamma(S',T',r')$ be two K-games. Γ' is called a "mixed extension of Γ" if S' is the set of all finite probability distributions over S, T' is the set of all finite probability distributions over T and $r'(s',t')$ is the expectation of the random variable $r(s,t)$ when s and t are distributed according to s' and t' respectively. Intuitively, Γ' is obtained from Γ by permitting the use of mixed strategies.

<u>Definition 12</u>. A K-game $\Gamma = \Gamma(S,T,r)$ is called "convex" is for every pair $s' \epsilon S$, $s'' \epsilon S$ and for every probability p, $0 < p < 1$, there is an $s \epsilon S$ such that $r(s,t) = (1-p)r(s',t) + pr(s'',t)$ for all $t \epsilon T$, and if for every pair $t' \epsilon T$, $t'' \epsilon T$ and for every p, $0 < p < 1$, there is a $t \epsilon T$ such that $r(s,t) = (1-p)r(s,t') + pr(s,t'')$ for all $s \epsilon S$.

<u>Theorem 4. Every convex K-game is equivalent to its mixed extension.</u>

The proof is straightforward, and is omitted.

<u>Example 3</u>. The mixed extension of "matching pennies" (example 1) is classical with the optimal strategy of the opposition $t^0 = 1/2$. Matching Pennies thus becomes classical by the introduction of mixed strategies.*

The next example shows that there are convex nonclear K-games.

<u>Example 4</u>. (See Fig. 2). Let

$$K = \{1,2\},\ S = \{s | 0 \leq s \leq 1\},\ T = \{t | 0 \leq t \leq 1\},$$

$$r(s,t) = (1-s)(1-t)(2,-1) + (1-s)t(-2,1) + s(1-t)(1,-2) + st(-1,2),$$

$$\Gamma_4 = \Gamma(S,T,r).$$

*In fact, if Γ is any K-game where K has only one member and S and T are both finite, then the mixed extension of Γ is classical. Indeed, such a game is essentially a finite two-person zero-sum game, and its classicality is equivalent to the minimax theorem.

Γ_4 is convex. It is the mixed extension of Γ_2. Its characteristic sets are

$$V(s) = \{f \mid f \leq (s-2, -1-s)\} ;$$

$$V = \{f \mid f \leq (-1,-1),\ f(1) + f(2) \leq -3\},$$

$$U(t) = \{f \mid f \leq (2-4t, 2t-1)\},\ \text{if } t \leq 1/2 ,$$

$$U(t) = \{f \mid f \leq (1-2t, 4t-2)\},\ \text{if } t \geq 1/2 ,$$

$$U = U(0) \quad U(1) = \{f \mid f \leq (-1,-1)\} .$$

Γ_4 is not pure. The opposition cannot prevent the payoff vector $(-1,-1)$, and the coalition cannot compel it.

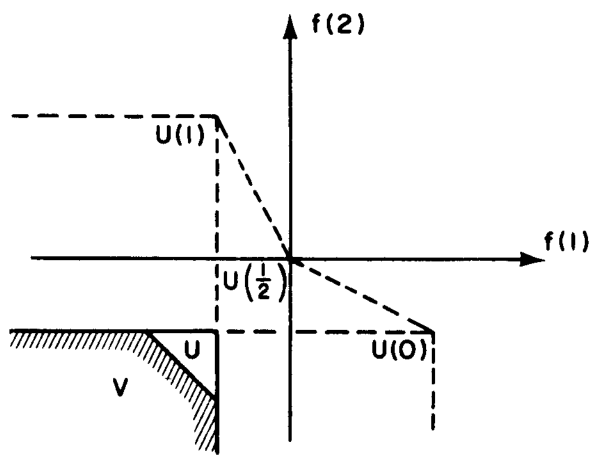

Fig. 2

1.3. Perfect Information

We have seen that in K–games the admission of mixed strategies was not sufficient to "clarify matters." Now we will show in two examples that perfect information does not play the same role in cooperative games that it does in two–person zero–sum games.

Example 5. Let

$K = \{1\}$, $S = \{0,1\}$, $T = \{00,01,10,11\}$,

$r(0,00) = r(0,01) = (2,-1)$, $r(0,10) = r(0,11) = (-2,1)$,

$r(1,00) = r(1,10) = (1,-2)$, $r(1,01) = r(1,11) = (-1,2)$,

$\Gamma_5 = \Gamma(S,T,r)$.

Γ_5 can be represented by the "tree" in Fig. 3. "K" indicates a move by the coalition and "O" indicates a move by the opposition. Γ_5 is of perfect information in the sense of von Neumann and Morgenstern [4]. The characteristic sets of Γ_5 are:

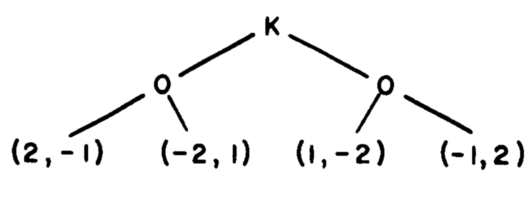

Fig. 3

$$V(0) = \{f \mid f \leq (-2,-1)\} ,$$

$$V(1) = \{f \mid f \leq (-1,-2)\} ,$$

$$V = V(0) \cup V(1),$$

$$U(00) = \{f \mid f \leq (2,-1)\},$$

$$U(01) = \{f \mid f \leq (2,-1) \text{ or } f \leq (-1,2)\},$$

$$U(10) = \{f \mid f \leq (-2,1) \text{ or } f \leq (1,-2)\},$$

$$U(11) = \{f \mid f \leq (-1,2)\},$$

$$U = U(00) \cap U(10) \cap U(11).$$

We have $U = V$, so Γ_5 is clear. But it is not classical; the opposition has no optimal strategy.

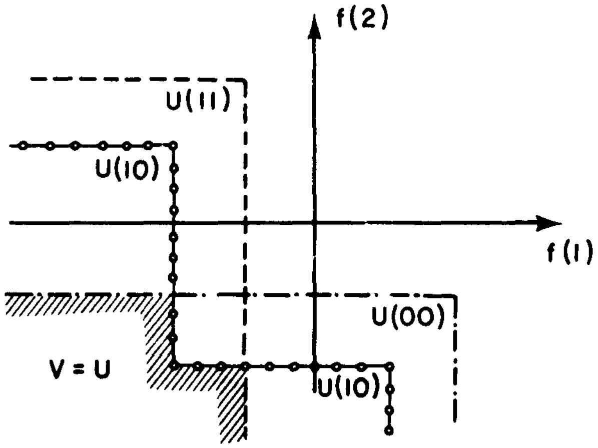

Fig. 4

Example 6. Let Γ_6 be the mixed extension of Γ_5 (example 5). Then it can be verified that (-1,-1) is in U but not in V; indeed if $f \in V$ and $f(1) = f(2)$, then $f \leq (-1.5,-1.5)$. Hence Γ_6 is not clear.

Theorem 5. A K-game of perfect information is not necessarily classical. There are nonclear K-games with perfect information. There are clear nonclassical K-games. The mixed extension of a clear K-game is not necessarily clear.

Proof. Examples 5 and 6.

In von Neumann and Morgenstern the distinction between clear and nonclear, and between classical and nonclassical games did not arise. In their theory it was assumed that the coalition was concerned only with the sum of the payoffs of the coalition members, and that the coalition could distribute this sum arbitrarily. How this should be done in particular (especially when the "payoffs" are really "expected payoffs") was not mentioned. In the next section we will investigate several examples of side payments in K-games.

1.4. Side Payments

In the examples studied so far the coalition members were not allowed to make side payments to each other. Now we wish to allow the coalition members to make side payments to each other in one form or another.

Example 7. Let

$K = \{1,2\}$, $S = \{s \mid s \text{ real}\}$, $T = \{f \mid 0 \leq t \leq 1\}$,

$r(s,t) = (1-t)(1-s,s) + t(-s,1+s)$,

$\Gamma_7 = \Gamma(S,T,r)$.

Coalition member 1 pays to member 2 the amount of s dollars (he receives $-s$ dollars from member 2 in case $s < 0$.) The opposition does not know this. It must pay one dollar to player 1 or 2, and may determine the recipient by a chance mechanism. The coalition members are guided by the expected payoff.

The characteristic sets of Γ_7 are

$V(s) = \{f \mid f \leq (-s,s)\}$,

$V = \{f \mid f(1) + f(2) \leq 0\}$,

$U(t) = \{f \mid f(1) + f(2) \leq 1\}$,

$U = \{f \mid f(1) + f(2) \leq 1\}$.

Γ_7 is not clear.

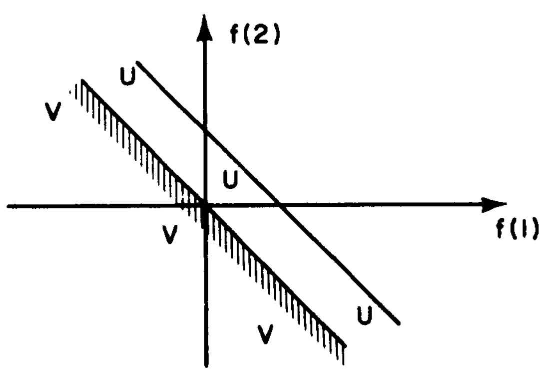

Fig. 5

<u>Example 8</u>. Let

$K = \{1,2\}$, $S = \{s = (s',s'') | s' \text{ real}, s'' \text{ real}\}$,

$T = \{t | 0 \leq t \leq 1\}$, $r(s,t) = (1-t)(1-s',s') + t(-s'',1+s'')$,

$\Gamma_8 = \Gamma(S,T,r)$.

Here again the opposition pays one dollar to one to the coalition members, and may choose the recipient by a chance mechanism. The coalition members make the following agreement: player 1 will pay s' dollars in case he receives the dollar from the opposition; he will pay s'' dollars in case the dollar goes to player 2. The opposition does not know of this agreement.

The characteristic sets of Γ_8 are:

$V(s) = \{f | f \leq (\text{Min}\{1-s',-s''\}, \text{Min}\{s',1+s''\})\}$,

$V = \{f | f(1) + f(2) \leq 1\}$,

$U(t) = U = V$.

Γ_8 is classical; every strategy of the opposition is optimal.

Examples 7 and 8 show that in general it is not the same whether the side-payments must be carried out before beginning the "actual game" (von Neumann and Morgenstern [4] only discuss examples in which this is the case) or whether the side-payments may have strings attached.

Example 9. Let

$K = \{1,2\}$, $S = \{s \mid 0 \leq s \leq 1000\}$,

$T = \{t \mid 0 \leq t \leq 1\}$, $0 < p < 1$,

$r(s,t) = (1-t)((1-p)(0,0) + p(-s,s)) + t(-s,s)$,

$\Gamma_9 = \Gamma(S,T,r)$.

The characteristic sets of Γ_9 are:

$V(s) = \{f \mid f \leq (-s, ps)\}$,

$V = \{f \mid f \leq (0,\ 1000p) \text{ and } pf(1) + f(2) \leq 0\}$,

$U(t) = \{f \mid f \leq (0,\ 1000((1-t)p + t)) \text{ and } f(1) + f(2) \leq 0\}$,

$U = U(0) = \{f \mid f \leq (0,\ 1000p) \text{ and } f(1) + f(2) \leq 0\}$.

Γ_9 is not clear. To be sure, every payoff vector that can be at all prevented by the opposition, can already be prevented by using the strategy $t = 0$; but $t = 0$ is not optimal. Figure 6 illustrates the case $p = 1/2$.

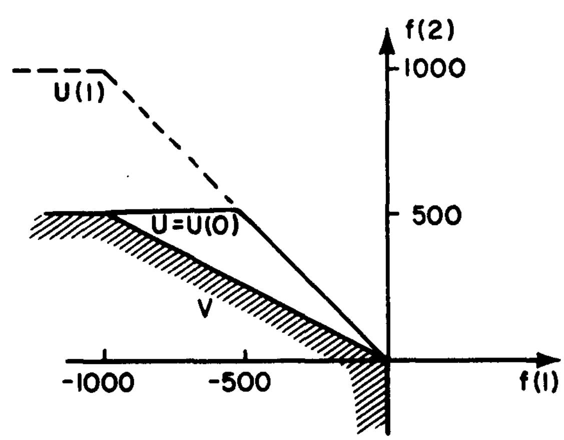

Fig. 6

Γ_9 can be interpreted as follows: the coalition members have no ready cash at the outset. However, player 1 has prospects of a bank loan in the amount of 1000 dollars. He promises s dollars to his coalition partner in case he should receive the loan. The bank makes inquiries about player 1 from the opposition. If the opposition gives a favorable report, then player 1 will get the credit; otherwise he will get it only with probability p.

The coalition knows the probability p. However it only will learn indirectly if the opposition has given an unfavorable report, and then only when credit is refused player 1. It is again assumed that the coalition members are guided by the expected payoff (player 1 naturally must subtract the bank debt from his account). Our example has been kept very simple. For instance, we have neglected interest. The availability of cash plays a large role in economic theory, as well as in practice. Our example shows that one cannot count on "clear" cooperative games as models for situations in which the availability of cash plays a role.

2. K-GAMES WHOSE PAYOFFS ARE CATALOGUES

2.1. Catalogues. K-Games Whose Payoffs Are Catalogues

Definition 13. Let K be a coalition, E = E(K). A subset F of E is called a "K-catalogue" or simply a "catalogue" if

F is nonempty, (K1),

F is convex, (K2),

and $f \leq f'$ and $f' \epsilon F$ imply $f \epsilon F$, (K3).

(K3) says that F contains all the vectors that it dominates.*

Example 10. For each real y, let

$$F(y) = \left\{f \epsilon E(K) \mid \Sigma_{k \epsilon K} f(k) \leq y\right\} .$$

We call F(y) a "von Neumann K-catalogue."

Definition 14. If

$$F_n \subset E,\ p_n \geq 0,\ F \neq 0 \text{ for } n = 1,\ldots,N,\ \Sigma_{n=1}^{N} p_n = 1,$$

then let

$$\Sigma_{n=1}^{N} p_n F_n = \left\{f \epsilon E \mid f = \Sigma_{n=1}^{N} p_n f_n, f_n \epsilon F_n \text{ for } n = 1,\ldots,N\right\} .$$

Theorem 6. If for $n = 1,\ldots,N$, F_n is a K-catalogue, and if

$$p_n \geq 0,\ \Sigma_{n=1}^{N} p_n = 1, \quad \Sigma_{n=1}^{N} p_n F_n = F,$$

then F is likewise a K-catalogue.

In words: a convex combination of K-catalogues is a K-catalogue. Intuitively, if the coalition can with probability p_n select an arbitrary payoff vector from the catalogue F_n, then it can certainly select an arbitrary payoff vector from the catalogue F.

The proof is straightforward and is omitted.

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*Stearns [6] calls this "comprehensiveness."

Definition 15. We call Γ a "K—game whose payoffs are catalogues", or a "K—game for catalogues", if M,N,L,W are nonempty finite sets,

$$z = (z_\ell)_{\ell\in L},\ z_\ell \geq 0 \text{ for } \ell\in L,\ \Sigma_{\ell\in L}\, z_\ell = 1.$$

d is a mapping of the cartesian product MxNxL into W (so that $d(m,n,l)\in W$ for all $m\in M$, $n\in N$, $\ell\in L$), F_w is a K—catalogue for all $w\in W$,

$$X = \left\{x \mid x = (x_m)_{m\in M},\ x_m \geq 0 \text{ for } m\in M,\ \Sigma_{m\in M}\, x_m = 1\right\},$$

$$S = \left\{s \mid s = (x,(f_{mw})_{m\in M,w\in W}),\ x\in X,\ f_{mw}\in F_w \text{ for } m\in M,\ w\in W\right\},$$

$$T = \left\{t \mid t = (t_n)_{n\in N},\ t_n \geq 0 \text{ for } n\in N,\ \Sigma_{n\in N}\, t_n = 1\right\},$$

$$r(s,t) = \Sigma_{m\in M}\Sigma_{n\in N}\Sigma_{\ell\in L}\, x_m t_n z_\ell f_{m,d(m,n,\ell)} \text{ and}$$

$$\Gamma = \Gamma(S,T,r).$$

We then also write $\Gamma = \Gamma(M,N,L,W,z,d,(F_w)_{w\in W})$.

In a K—game for catalogues, the coalition selects its pure fighting strategies $m\in M$ and its distribution methods* $(f_{mw})_{w\in W}$ with probabilities x_m, where $f_{mw}\in F_w$ for $w\in W$. The opposition independently chooses one of its pure strategies n with probability t_n. Chance selects one of its "strategies" ℓ with probability z_ℓ. If the fighting strategy m and strategies n and ℓ are used, a situation w results in which the coalition is entitled to choose one

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*A "strategy" in the sense of definition 1 is a pair consisting of a "fighting strategy" and a "distribution method."

payoff vector from the catalogue F_w. According to plan it selects f_{mw}. Thus the vector $f_{m,d(m,n,\ell)}$ results with probability $x_m t_n z_\ell$.

Theorem 7. K-games for catalogues are convex.

The proof is straightforward, and is omitted.

Editor's Note: K-games for catalogues are a natural generalization of K-games with unrestricted side payments. In the latter, strategies (more precisely, "fighting strategies") x and t are chosen by the coalition and opposition respectively; the resulting total payoff v(x,t) can then be divided in an arbitrary way by the coalition. Conceivably, the coalition could even throw away part of the total payoff. This means that the actual result of the strategies (x,t) is the von Neumann K-catalogue

$$F = \left\{f: \Sigma_{k\in K} f(k) \leq v(x,t)\right\}$$

from which the coalition may choose any payoff vector it pleases (see Fig. 7).

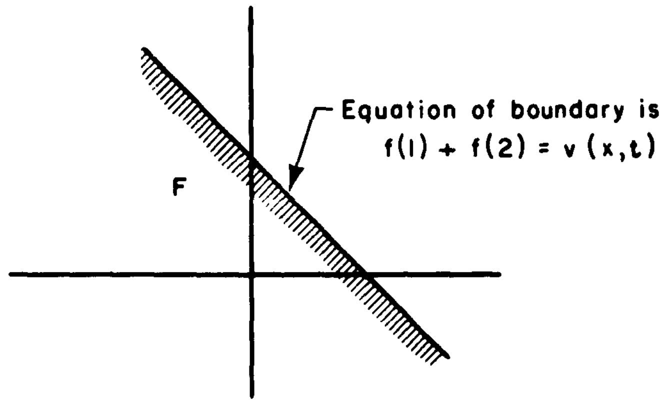

Fig. 7

In a K-game for catalogues, the payoff to a strategy pair (x,t) is not necessarily a von Neumann catalogue, but may be an arbitrary catalogue. This could result from a situation in which side payments are permitted but somehow restricted, or side payments are unrestricted in monetary terms but utility is not linear in money (see example 11 below).

Briefly: A K-game for catalogues is a finite two-person game in matrix form, in which the entries of the matrix are K-catalogues rather than numbers. The two "players" of the game are the coalition and the opposition. After a play is completed, the coalition picks a payoff vector from the catalogue that resulted from the play. Provision is also made for chance moves.

Example 11. If

$$y > 0,\ B(y) = \left\{f \in E(K) \mid \Sigma_{k\in K} \exp(f(k)) \leq y\right\},$$

then we call B(y) "Bernoullian."

Explanation: An amount y is divided among the coalition members. Following Bernoulli, the members have logarithmic utility functions. Assuming an appropriate choice of scale, coalition member k therefore gets the utility log(y(k)) from the portion $y(k) > 0$. From $\Sigma_{k\in K} y(k) \leq y$ and $f(k) = \log(y(k))$ it follows that $\Sigma_{k\in K} \exp(f(k)) \leq y$. B(y) is convex. Intuitively, B(y) is the set of payoff vectors that are accessible to the coalition if it has the amount y to divide.

2.2. K-Games For Catalogues From $\mathfrak{F}$

In the following, K will be an arbitrary but fixed coalition. When sets of catalogues are mentioned, this will mean that the catalogues are all K-catalogues for the same K.

Definition 16. If $\mathfrak{F}$ is a nonempty set of catalogues, and $\Gamma = \Gamma(M,N,L,W,z,d,\ (F_w)_{w\in W})$ is a K-game for catalogues which has the property that $F_w \in \mathfrak{F}$ for all $w \in W$, then we call Γ a K-game "for catalogues from $\mathfrak{F}$."

Example 12. Let $\mathfrak{F}_1$ be the set of all von Neumann catalogues F(y). We call a K-game for catalogues from $\mathfrak{F}_1$ a "von Neumann K-game."

We will see later that von Neumann K-games are always classical. In order to demonstrate this and more general results, we need some more definitions and preliminary theorems.

Definition 17. A set $\mathfrak{F}$ of catalogues is called "clear" if every K-game for catalogues from $\mathfrak{F}$ is clear. A set $\mathfrak{F}$ of catalogues is called "classical" if every K-game for catalogues from $\mathfrak{F}$ is classical.

Theorem 8. Every classical set of catalogues is clear.

Proof: Follows from Theorem 2.

Example 13. We stated above that the set $\mathfrak{F}_1$ of von Neumann catalogues is classical, and that this would be demonstrated in the sequel. It can be shown that the set $\mathfrak{F}_2$ of Benoullian catalogues B(y) (example 11) is also classical.

Definition 18. If $\mathfrak{F}$ is a nonempty set of catalogues, then

$$CH(\mathfrak{F}) = \left\{F \subset E(K) \mid F = \Sigma_{n=1}^{N} p_n F_n,\ p_n \geq 0,\ F_n \epsilon \mathfrak{F},\ \Sigma_{n=1}^{N} p_n = 1\right\}.$$

We call CH($\mathfrak{F}$) the "convex hull" of $\mathfrak{F}$.

Theorem 9. CH($\mathfrak{F}$) *consists of catalogues.*

Proof: Theorem 6.

Theorem 10. If $\mathfrak{F}$ *is a set of catalogues,* Γ *a K-game for catalogues from* CH($\mathfrak{F}$), *then there is a K-game for catalogues from* $\mathfrak{F}$ *which is equivalent to* Γ.

The idea of the proof is as follows: each strategy triple (opposition, coalition, chance) in Γ yields a catalogue $\mathfrak{F}$ in CH($\mathfrak{F}$). The catalogue $\mathfrak{F}$ is a probability combination of catalogues $\mathfrak{F}_n$ in $\mathfrak{F}$, say with probabilities p_n. If instead of awarding $\mathfrak{F}$ for the given strategy triple, we allow chance to choose $\mathfrak{F}_n$ with probability p_n and then award $\mathfrak{F}_n$, we get the desired K-game for catalogues from $\mathfrak{F}$ that is equivalent to Γ. Details of the proof are left to the reader.

Theorem 11. If a set $\mathfrak{F}$ *of catalogues is clear, then* CH($\mathfrak{F}$) *is also clear. If a set* $\mathfrak{F}$ *of catalogues is classical, then* CH($\mathfrak{F}$) *is also classical.*

Proof: Follows from Theorems 3 and 10.

Definition 19. Let $\mathfrak{F}$ be a nonempty finite set of catalogues,

$$F_0 = \bigcap_{F \in \mathfrak{F}} F, \; F_1 \text{ the convex hull of } \bigcup_{F \in \mathfrak{F}} F,$$

$$F_p = (1-p)F_0 + pF_1 \text{ for } 0 \leq p \leq 1.$$

$\mathfrak{F}$ is called r.o. ("real-ordered", i.e., ordered like the reals) if for every $F \in \mathfrak{F}$ there is a p such that $0 \leq p \leq 1$ and

$$F = F_p, \qquad \text{(r.o. 1)}$$

and if for every $f \in F_p$ there is an $f_0 \in F_0$ and an $f_1 \in F_1$ such that

$$f = (1-p)f_0 + pf_1 \text{ and } f_0 \leq f \leq f_1. \qquad \text{(r.o. 2)}$$

A set of catalogues is called "r.o." if every finite nonempty subset of catalogues is r.o.

<u>Editor's Note</u>: That (r.o. 2) does not follow from (r.o. 1) can be seen by the example sketched in Fig. 8. If we define $\mathfrak{F} = \left\{F_p = (1-p)F_0 + pF_1 \,|\, 0 \leq p \leq 1\right\}$, then

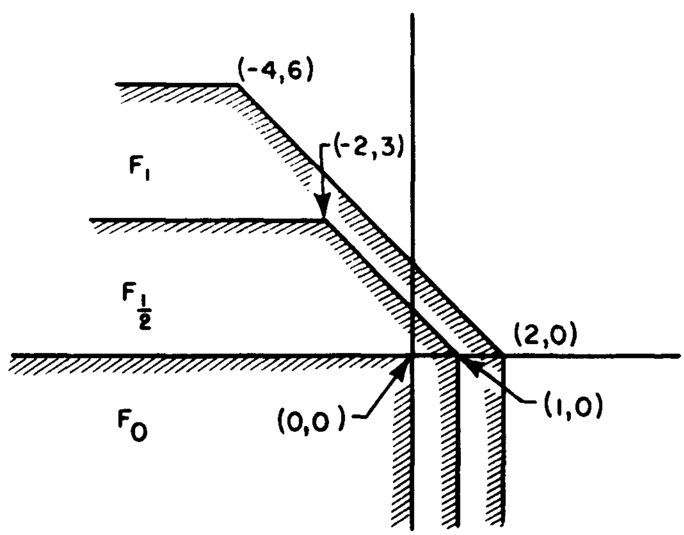

Fig. 8

$\mathfrak{F}$ satisfies (r.o. 1) but not (r.o. 2);
indeed $(-2,3) \in F_{1/2}$, but there do not
exist $f_0 \in F_0$ and $f_1 \in F_1$ such that $f_0 \leq (-2,3) \leq f_1$
and $f = 1/2\ f_0 + 1/2\ f_1$.

Theorem 12. $\mathfrak{F}_1$, the set of von Neumann catalogues (see example 10), is r.o.

This is geometrically evident. The formal proof is left to the reader.

Theorem 13. Every r.o. set of catalogues is classical.

The idea of the proof is as follows: Because $\mathfrak{F}$ is r.o., it is possible to associate with each F in $\mathfrak{F}$ a real "index"* α such that $(1-p)F_\alpha + pF_\beta = F_{(1-p)\alpha + p\beta}$, and $F_\alpha \supset F_\beta$ if and only if $\alpha \geq \beta$. The coalition is therefore interested in playing in such a way as to maximize the expected value of the index and the opposition in minimizing it. Thus we have essentially a finite** two-person 0-sum game, and such games are classical.*** The details of the proof are left to the reader.

Corollary. The set of von Neumann catalogues is classical. Every von Neumann K-game is classical.

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*This is the "social utility function" mentioned in the introduction.

**Finiteness is specified in definition 15.

***See footnote on page 14.

Proof: Theorems 12 and 13.

Theorem 13 does not provide us with all of the classical sets of catalogues. This is illustrated by the following example.

Example 14. Let

$K = \{1,2\}$,

$F = \{f \mid f(1) + f(2) \leq 1,\ f(1) \leq 1,\ f(2) < 1\}$,

$H = \{f \mid f(1) + f(2) \leq 1,\ f(1) < 1,\ f(2) \leq 1\}$.

$\mathfrak{I}_3 = \{F,H\}$.

$\mathfrak{I}_3$ is obviously not r.o. ((r.o. 1) is false). It is, however, classical; indeed any mixed strategy of the opposition that utilizes every pure strategy with positive probability is optimal.

The converse of Theorem 8 is also false. There are clear sets of catalogues that are not classical.

Example 15. $K = \{1,2\}$,

$\mathfrak{I}_4 = \{\{f \mid f(1) \leq 0\}, \{f \mid f(2) \leq 0\}\}$.

$\mathfrak{I}_4$ is clear but not classical. Four cases can arise:

$V = U = \{f \mid f \leq (0,0)\}$.

$V = U = \{f \mid f(1) \leq 0\}$,

$V = U = \{f \mid f(2) \leq 0\}$,

$V = U = E(\{1,2\})$.

Example 15 shows that the definition of catalogue admits quite pathological cases. In the following section we will introduce the concept of a "regular" catalogue. The converses of Theorems 8 and 13 are valid for regular catalogues.

Our final example is:

Example 16. It can be shown directly that the set $\mathfrak{J}_2$ of Bernoullian catalogues is r.o., from which it follows by Theorem 13 that it is classical (as remarked in example 13). In addition to the Bernoulli utility functions there is a family of other utility functions (obtained as solutions of a differential equation of 3rd order with one parameter) that generate r.o. and therefore classical sets of catalogues. We will not deal with them here.*

3. K-GAMES FOR REGULAR CATALOGUES

3.1. Regular Catalogues

Inner products, the norm, and limits in E are as defined as usual; that is, $f.h = \Sigma_{k\epsilon K} f(k)h(k)$, $|f| = \sqrt{f.f}$, and $\lim f_n = f$ if and only if $\lim f_n(k) = f(k)$ for all k (or equivalently, $\lim |f - f_n| = 0$). Furthermore, we agree that $f > 0$ shall mean $f(k) > 0$ for all $k\epsilon K$.

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*On the other side of the ledger, it is easy to give examples of utility functions that generate sets of catalogues that are neither r.o. nor clear nor classical.

Definition 20. Let $u \epsilon E$, $u > 0$ and $F \subset E$, $F \neq 0$. Then we define $u(F) = \sup_{f \epsilon F} u.f$, and $F/u = \{f \epsilon F | u.f = u(F)\}$.

Definition 21. A catalogue F is called "regular" if for all $u \epsilon E$ such that $u > 0$ we have

$u(F) < \infty$, (K4)

F is closed. (K5)

Example 17. If $f_n \epsilon E$ for $n = 1,\ldots,N$, and

$$F(f_1,\ldots,f_N) = \left\{f \epsilon E | f \leq \Sigma_{n=1}^N p_n f_n,\ p_n \geq 0,\ \Sigma_{n=1}^N p_n = 1\right\},$$

then we say that F is "generated" by the payoff vectors $f_1,\ldots,f_n$.

Example 18. If F is a catalogue, $f \epsilon E$, and for all $f' \epsilon F$ we have $f' \leq f$, then we say F is "bounded" (short for "bounded from above"). Catalogues generated by a finite number of payoff vectors are bounded. Bounded closed catalogues are regular. Catalogues generated by a finite number of payoff vectors are regular.*

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*Geometrically, a regular catalogue F is one which has a supporting hyperplane in each strictly positive direction. That is, for each $u > 0$ there is hyperplane perpendicular to u that supports F. Von Neumann catalogues are not regular. Bernoullian catalogues are regular. In general, catalogues generated by utility functions v_k of the coalition members are regular if $v_k(y)/y \to 0$ as $y \to \infty$.

<u>Lemma 1. If</u> F <u>is a regular catalogue,</u> $f_n \epsilon F$ $n = 1,2,\ldots$ <u>an arbitrary sequence of payoff vectors in</u> F, <u>then at least one of the following two cases applies</u>:

<u>Case 1: For some subsequence</u> (f_{nr}) <u>of</u> (f_n), $\lim f_{nr} = f \epsilon F$.

<u>Case 2: For some subsequence</u> (f_{nr}), $\lim |f_{nr}| = +\infty$, $\lim |f_{nr}|^{-1} f_{nr} = e \epsilon E$ <u>and</u> $|e| = 1$, $e \leq 0$.

<u>Proof</u>: The sequence (f_n) is either bounded or unbounded (in norm).

Case 1: If the sequence (f_n) is bounded then it contains a convergent subsequence; this converges to an element of F, because F is closed.

Case 2: If the sequence (f_n) is unbounded then it contains a subsequence whose sequence of norms tends to infinity. If we divide every element of this sequence by its norm, we will obtain a sequence of unit vectors. This is bounded and therefore contains a convergent subsequence. The set of unit vectors is closed; the unit vectors therefore converge to a unit vector. Now it remains to be shown that for this unit vector e we have $e \leq 0$, i.e., $e(k) \leq 0$ for all $k \epsilon K$.

Let us assume there is a $k'' \epsilon K$ such that $e(k'') > 0$. Define $u \epsilon E$ by the expression

$$u(k') = 1 \text{ for } k' \neq k'',\ u(k'') = (e(k''))^{-1}(|K| + 1),$$

where $|K|$ is the number of coalition partners. Then

$$u.e = u(k'')e(k'') + \Sigma_{k' \neq k''} u(k')e(k')$$

and because $e(k') \geq -1$, $u.e \geq (|K| + 1) - (|K| - 1) = 2$. Because $\lim |f_{nr}|^{-1} f_{nr} = e$, for sufficiently large n_r we have $u.(|f_{nr}|^{-1} f_{nr}) \geq 1$. Then it follows that $u.f_{nr} \geq |f_{nr}|$ for sufficiently large n_r and since $(f_{nr}) \subset F$, $\lim |f_{nr}| = +\infty$, it follows that $u(F) = +\infty$. But since $u > 0$ this contradicts the definition of a regular catalogue. This completes the proof.

Theorem 14. *If* F *is a regular catalogue,* $u \epsilon E$, *and* $u > 0$, *then* F/u *is a nonempty, bounded, closed and convex set.*

Proof: There is a sequence of payoff vectors f_n such that $f_n \epsilon F$ and $\lim u.f_n = u(F)$. Let us apply Lemma 1 to this sequence. Suppose Case 2 applies; we then have $\lim |f_{nr}|^{-1} = 0$, $\lim u.f_{nr} = u(F) < \infty$, $\lim |f_{nr}|^{-1} f_{nr} = e$ with $|e| = 1$ and $e \leq 0$. Then $\lim |f_{nr}|^{-1} u.f_{nr} = 0$ and $\lim u.(|f_{nr}|^{-1} f_{nr}) = u.e < 0$, a contradiction. Therefore, Case 1 must apply: $\lim f_{nr} = f \epsilon F$. For this f, $u.f = \lim u.f_{nr}$ $= u(F)$, therefore $f \epsilon F/u$; so F/u is nonempty. Next, F/u must also be bounded; otherwise there would be a sequence f_n in F/u which contains no convergent subsequence. Lemma 1 applied to this sequence would result in $u.f_{nr} = u(F)$, $\lim |f_{nr}|^{-1} = 0$, therefore $\lim u.(|f_{nr}|^{-1} f_{nr}) = 0$ and $\lim |f_{nr}| u.f_{nr} = u.e < 0$, a contradiction. Finally, as the intersection of the convex and closed set F and the convex and closed hyperplane $\{f|u.f = u(f)\}$, F/u is closed and convex.

<u>Lemma 2. If</u> F <u>is a regular catalogue,</u> $h\epsilon E$, <u>and</u> $u.h \leq u(F)$ <u>for all</u> $u\epsilon E$ <u>such that</u> $u > 0$, <u>then</u> $h\epsilon F$.

<u>Proof</u>: Since F is nonempty and closed (K5), there is an $f\epsilon F$ such that

$$(h-f).(h-f) \leq (h-f').(h-f') \text{ for all } f'\epsilon F. \tag{1}$$

F is convex. Therefore for the same f, the following also holds:

$$(h-f).f = \sup_{f'\epsilon F} (h-f).f'. \tag{2}$$

Because of (K3), $f \leq h$, i.e.,

$$(h-f) \geq 0; \tag{3}$$

for otherwise we could define f' by $f'(k) = \text{Min}\{f(k),h(k)\}$. Then we would have $f\epsilon F$ and $f' \leq f$, and so according to (K3), $f'\epsilon F$. Hence

$$(h-f').(h-f') = (h-f).(h-f) - \Sigma_{f(k) > h(k)}(f(k)-h(k))^2,$$

contradicting (1); this establishes (3). Next, select an arbitrary but fixed $u\epsilon E$ with $u > 0$. For real $c > 0$, set $u_c = h - f + cu$. Certainly $u_c\epsilon E$ and $u_c > 0$, therefore $h.u_c \leq u_c(F)$, by hypothesis. Therefore

$$\begin{aligned}(h-f).h + cu.h &\leq \sup_{f'\epsilon F} ((h-f).f' + cu.f') \\ &\leq \sup_{f'\epsilon F} (h-f).f' + c\sup_{f''\epsilon F}u.f''.\end{aligned}$$

Application of (2) and Definition 20 yields

$$(h-f).h + cu.h \leq (h-f).f + cu(F), \text{ i.e.,}$$

$$(h-f).(h-f) \leq c(u(F) - u.h) \text{ for all } c > 0.$$

Since $u(F) - u.h$ is finite, this is only possible if $(h-f).(h-f) = 0$; therefore $h = f$. This proves $h \epsilon F$.

Theorem 15. If F and H are two regular catalogues, $0 \leq p \leq 1$, $G = (1-p)F + pH$, then G is a regular catalogue.

Proof: It follows from Theorem 6 that G is a catalogue. We must establish the fact that G has the properties (K4) and (K5) in Definition 21.

Let $u \epsilon E$, $u > 0$. Then $u(G) = (1-p)u(F) + pu(H)$. Therefore G has the property (K4). Now we must show that G is closed (K5). For $p = 0$ ($G = F$) and $p = 1$ ($G = H$), there is nothing to prove. So let $0 < p < 1$. If $g_n \epsilon G$ for $n = 1,2,\ldots$, and $\lim g_n = g$, then there are sequences $f_n \epsilon F$, $h_n \epsilon H$, $n = 1,2,\ldots$ such that for all n, $g_n = (1-p)f_n + ph_n$. We apply Lemma 1 to the sequence (f_n).

Case 1: $\lim f_{nr} = f \epsilon F$. Because $h_{nr} = p^{-1}(g_{nr} - (1-p)f_{nr})$, there is an $h \epsilon H$ such that $\lim h_{nr} = h = p^{-1}(g - (1-p)f)$; therefore $g = (1-p)f + ph$, $g \epsilon G$.

Case 2 cannot occur. For if

$$\lim |f_{nr}| = +\infty, \lim |f_{nr}|^{-1} f_{nr} = e \leq 0,$$

then we would have

$$\lim |h_{nr}| = +\infty \text{ and } \lim |h_{nr}|^{-1} h_{nr} = -e \geq 0$$

(p is different from zero and one and the sequence (g_{nr}) is convergent, therefore bounded.) However, Lemma 1 applied to the sequence (h_{nr}) would give the result that $\lim |h_{nr}|^{-1} h_{nr} \leq 0$, since (h_{nr}) can contain no convergent subsequence. Since the requirements $|e| = 1$, $e \leq 0$, $-e \leq 0$ are irreconcilable, the proof is complete.

Theorem 16. If $\mathfrak{F}$ is a set of regular catalogues, then $CH(\mathfrak{F})$ is also a set of regular catalogues.

Proof: Follows from Theorem 15.

Lemma 3. If F and H are two regular catalogues, and if for $g \epsilon E$ we set $P(g) = \{p | 0 \leq p \leq 1,$ $g = (1-p)f + ph,\ f \epsilon F,\ h \epsilon H\}$, then $P(g)$ is convex and closed.

Proof: Convexity is straightforward. To prove closedness, let $p_n \epsilon P(g)$ for $n = 1,2,\ldots$, $\lim p_n = p$. There are sequences $f_n \epsilon F$, $h_n \epsilon H$ such that for $n = 1,2,\ldots$

$$g = (1-p_n)f_n + p_n h_n. \tag{1}$$

Let us apply Lemma 1 to the sequence f_n.

$$\text{Case 1:} \quad \lim f_{nr} = f,\ f \epsilon F \tag{2}$$

We again apply Lemma 1 to the appropriate sequence $\{h_{nr}\}$.

Case 1.1. The sequence $\{h_{nr}\}$ has a convergent subsequence. Then we have

$$\lim p_{nr\mu} = p, \ \lim f_{nr\mu} = f, \ \lim h_{nr\mu} = h \epsilon H;$$

passing to the limit, we obtain $g = (1-p)f + ph$, $f \epsilon F$, $h \epsilon H$, and so $p \epsilon P(g)$.

Case 1.2: $\lim f_{nr\mu} = f \epsilon F$, $\lim |h_{nr\mu}| = +\infty$, $\lim |h_{nr\mu}|^{-1} h_{nr\mu} = e \leq 0$, $|e| = 1$. Because of (1) and (2), this is only possible if $\lim p_n = p = 0$. If we show that $g \epsilon F$, then we will have shown that $p \epsilon P(g)$. Now because of (1) and $\lim p_{nr\mu} = 0$ and (2),

$$g - f = \lim (p_{nr\mu} h_{nr\mu}) = \lim (p_{nr\mu} |h_{nr\mu}|)(|h_{nr\mu}|^{-1} h_{nr\mu})$$

$$= |g - f| e \leq 0,$$

therefore $g \leq f$, $f \epsilon F$; so according to (K3), $g \epsilon F$.

Case 2: $\lim |f_{nr}| = +\infty$, $\lim |f_{nr}|^{-1} f_{nr} = e$, $|e| = 1$, $e \leq 0$. Again we will apply Lemma 1 to the corresponding sequence $\{h_{nr}\}$.

Case 2.1 (a subsequence of h_{nr} converges) is symmetric to Case 1.2; one proves that $p = 1$ and $g \epsilon H$, from which it follows that $p \epsilon P(g)$.

Case 2.2: $\lim |f_{nr\mu}| = \infty$, $\lim |h_{nr\mu}| = \infty$, $\lim |f_{nr\mu}|^{-1} f_{nr\mu} = e_f$, $\lim |h_{nr\mu}|^{-1} h_{nr\mu} = e_h$, $e_f \leq 0$, $e_h \leq 0$. If for $n > n_0$ it were always true that $0 < p_n < 1$, then it would be true for $n_{r\mu} > n_0$ that $|g - f_{nr\mu}|^{-1} (g - f_{nr\mu}) = |h_{nr\mu} - g|^{-1} (h_{nr\mu} - g)$. In the limit we would have $e_h = -e_f$, and this leads to the contradiction $|e_f| = 1$, $e_f \geq 0$, $e_f \leq 0$. It follows that

$p_n = 0$ for infinitely many n, or $p_n = 1$ for infinitely many n; then, however, $\lim p_n = 0$, or, respectively, $\lim p_n = 1$. In each case, though, $p_n = p$ for certain n. Therefore $p \in P(g)$, and the proof is complete.

Note: By repeated application of Lemma 1, we have used both the fact that the catalogues F and H are closed, as well as their property (K4). Lemma 3 is not valid for closed nonregular catalogues.

3.2. Clear Sets of Regular Catalogues

Theorem 17. *If $\mathfrak{F}$ is a clear set of regular catalogues and if F and H are in $\mathfrak{F}$, then $F \cup H$ is convex.*

Proof: For $g \in E$ let P(g) be as in Lemma 3. $P(g) \neq \emptyset$ if and only if g is an element of the convex hull of $F \cup H$. Further, $0 \in P(g)$ if and only if $g \in F$, and $1 \in P(g)$ if and only if $g \in H$. We must show that $0 \in P(g)$ or $1 \in P(g)$ if $P(g) \neq \emptyset$. Let $P(g) \neq \emptyset$. According to Lemma 3, P(g) is convex and closed. Therefore, there are probabilities p', p'', $0 \leq p' \leq p'' \leq 1$ such that $P(g) = \{p \mid p' \leq p \leq p''\}$. Let $c = \mathrm{Min}\{p', 1-p''\}$. If $c = 0$, then $p' = 0$ or $p'' = 1$, and $0 \in P(g)$ or $1 \in P(g)$. Therefore it is sufficient to show that $c = 0$.

As an aid let us define a K-game Γ_g for catalogues from $CH(\mathfrak{F})$.

$\Gamma_g = \Gamma(M,N,L,W,z,d,(F_w)_{w\epsilon W})$, where

$M = \{0,1\}$, $N = \{0,1\}$, $L = \{0\}$, $W = \{00,01,10,11\}$,

$d(m,n,0) = mn$,

$F_{00} = (1-p'+c)F + (p'-c)H$, $F_{01} = (1-p'')F + p''H$

$F_{10} = (1-p')F + p'H$, $F_{11} = (1-p''-c)F + (p''+c)H$

Then

$$X = \{(x_0, x_1) | x_0 \geq 0,\ x_1 \geq 0,\ x_0 + x_1 = 1\},$$

$$T = \{t = (t_0, t_1) |\ t_0 \geq 0,\ t_1 \geq 0,\ t_0 + t_1 = 1\},$$

$$S = \{s = (x_0, x_1, (f_{mw})_{m\epsilon M, w\epsilon W}), (x_0,x_1)\epsilon X,\ f_{mw}\epsilon F_w\},$$

and if we simply write f_{mn} in place of $f_{m,d(m,n,0)} = f_{m,mn}$, then

$$r(s,t) = x_0t_0f_{00} + x_0t_1f_{01} + x_1t_0f_{10} + x_1t_1f_{11}.$$

Choose $t\epsilon T$ arbitrarily, $t = (t_0,t_1)$. We will now define $s\epsilon S$ so that $x_0 = t_1$, $x_1 = t_0$,

$f_{00} = (1-p'+c)f + (p'-c)h$, $f_{01} = (1-p'')f + p''h$,

$f_{10} = (1-p')f + p'h$, $f_{11} = (1-p''-c)f + (p''+c)h$,

where $f\epsilon F$, $h\epsilon H$. Then for this t and the s we have just defined, we have $r(s,t) = (1-p)f + ph$, where $p = t_0p' + t_1p''$. Because of the convexity of $P(g)$, we have $p\epsilon P(g)$. We can therefore select $f\epsilon F$ and $h\epsilon H$ such that $(1-p)f + ph = g$. Thus for every $t\epsilon T$ there is an $s\epsilon S$ such that $r(s,t) = g$. Hence

$$g\epsilon U, \qquad (1)$$

where U is as defined in definition 7.

$\mathfrak{F}$ was assumed to be clear. According to Theorem 11, CH($\mathfrak{F}$) is also clear. Γ_g is a K-game for catalogues from CH($\mathfrak{F}$). Therefore V = U must be true, and so by (1), $g \in V$. So there is an $s \in S$ such that for all $t \in T$ we have $r(s,t) \geq g$. For this s there are two possibilities: 1. $x_0 = 1$, 2. $x_0 < 1$.

1. $x_0 = 1$, $x_1 = 0$. For $t' = (1,0)$, we must have $r(s,t') \geq g$. But this means $f_{00} \geq g$. So there must be an $f \in F$ and an $h \in H$ such that $f_{00} = (1-p' + c)f + (p' - c)h \geq g$. Because of the property (K3) of the catalogues F and H, there are even $f \in F$ and $h \in H$ such that $(1-p' + c)f + (p' - c)h = g$. Therefore $(p' - c) \in P(g) = \{p | p' \leq p \leq p''\}$. Because $c \geq 0$, this is only possible if $c = 0$.

2. $x_0 < 1$. For $t'' = (0,1)$, we have $r(s,t'') \geq g$. So $r(s,t'') = x_0 f_{01} + x_1 f_{11} \geq g$. Hence there must be elements $f_0 \in F$, $f_1 \in F$, $h_0 \in H$, $h_1 \in H$ such that $x_0(1-p'')f_0 + x_1(1-p''- c)f_1 + x_0 p'' h_0 + x_1(p'' + c)h_1 \geq g$. Because of the convexity of F and H, then, there are also elements $f \in F$ and $h \in H$ such that $(1-p''- x_1 c)f + (p'' + c)h \geq g$. As above, we conclude from this that $(p''+ x_1 c) \in P(g)$, from which because of $x_1 > 0$ it follows that $c = 0$. Thus it has been shown in each case that we must have $c = 0$.

Theorem 18. *If* $\mathfrak{F}$ *is a clear set of regular catalogues,* $F \in \mathfrak{F}$, $H \in \mathfrak{F}$, $f' \in F$, $h' \in H$, $0 < p < 1$, $g = (1-p)f' + ph'$, *then*

<u>there are elements</u> $f \epsilon F$, $h \epsilon H$ <u>such that</u> $(1-p)f + ph = g$ <u>and</u> <u>moreover, either</u> $f \leq g \leq h$ or $h \leq g \leq f$.

<u>Proof</u>. By the given assumptions, g is an element of the convex hull of $F \cup H$. According to Theorem 17, $F \cup H$ is itself convex, therefore $g \epsilon F \cup H$. There are three possible cases: 1. $g \epsilon F$ and $g \epsilon H$, 2. $g \epsilon F$ and $g \notin H$, 3. $g \notin F$ and $g \epsilon H$.

1. If $g \epsilon F$ and $g \epsilon H$, then we can select $f = g = h$.

2. This case is symmetric to the third case.

Thus the assertion in case 3 remains to be proved. Assume $g \notin F$, $g \epsilon H$. We define a K-game with catalogues from $\mathfrak{F}$ as follows:

$$\Gamma_g' = \Gamma(M,N,L,W,z,d,(F_w)_{w \epsilon W}) \text{ with } M = \{0\},\ N = \{0,1\},$$

$$L = \{0,1\},\ W = \{0,1\},\ z = (z_0,z_1) = (1-p,p),$$

$$d(0,0,0) = 0,\ d(0,n,\ell) = 1 \text{ for } (n,\ell) \neq (0,0),$$

$$F_0 = F,\ F_1 = H.$$

In simplified notation we have the following:

$$S = \{s \mid s = (f,h),\ f \epsilon F,\ h \epsilon H\},$$

$$T = \{t \mid t = (t_0,t_1),\ t_0 \geq 0,\ t_1 \geq 0,\ t_0 + t_1 = 1\},$$

$$r(s,t) = t_0(1-p)f + (t_0 p + t_1)h.$$

Choose $t \epsilon T$ arbitrarily, $t = (t_0,t_1)$. Because $p > 0$ and $t \epsilon T$, $(t_0 p + t_1) > 0$. Because $h' \epsilon H$ and $g \epsilon H$, since H is convex, $(t_0p + t)^{-1}(t_0ph' + t_1g) = h''\epsilon H$. If we set $s = (f', h'')$, then $s\epsilon S$ and $r(s,t) =$

$$r(s,t) = t_0(1-p)f' + (t_0p + t_1)(t_0p + t_1)^{-1}(t_0ph' + t_1g)$$

$$= t_0((1-p)f' + ph') + t_1g.$$

It was assumed that $g = (1-p)f' + ph'$, therefore $r(s,t) = g$. So for every $t\epsilon T$ there is an $s\epsilon S$ such that $r(s,t) = g$. Therefore $g\epsilon U$ (where U is as in Definition 7).

$\mathfrak{F}$ was assumed to be clear, therefore the K-game Γ'_g for catalogues from $\mathfrak{F}$ must be clear. Since $g\epsilon U$ it follows that $g\epsilon V$, so $g\epsilon V(s)$ for some $s\epsilon S$. So there must be an $f\epsilon F$ and an $h\epsilon H$ such that for all $t\epsilon T$, we have $r((\tilde{f},h),t) \geq g$. In particular this inequality must hold for $t = (0,1)$ and for $t = (1,0)$. The following inequalities result: $h \geq g$ and $(1-p)\tilde{f} + ph \geq g$. If we set $f = (1-p)^{-1}(g-ph)$, then because of $\tilde{f} \geq f$ and (K3) for F, we deduce $\tilde{f} \geq f$. So $h \geq g \geq f$ and $(1-p)f + ph = g$.

<u>Lemma 4. If</u> $\mathfrak{F}$ <u>is a clear set of regular catalogues,</u> $F\epsilon\mathfrak{F}$, $H\epsilon\mathfrak{F}$, $u\epsilon E(K)$, $u > 0$, $u(F) \leq u(H)$, <u>then there is an</u> $f\epsilon F/u$ <u>and an</u> $h\epsilon H/u$ <u>such that</u> $f \leq h$.

<u>Proof</u>: By Theorem 14, F/u and H/u are nonempty. Therefore there are $f'\epsilon F/u$ and $h'\epsilon H/u$. We will select a p, $0 < p < 1$, set $g = (1-p)f' + ph'$ and apply Theorem 18. As a result we obtain an $f\epsilon F$ and an $h\epsilon H$ such that $g = (1-p)f + ph$ and $f \leq g \leq h$ or $h \leq g \leq f$. Since

$f' \epsilon F/u$, $h' \epsilon H/u$, therefore $(1-p)u.f + pu.h = u.g = (1-p)u(F) + pu(H)$. Because of $0 < p < 1$ and the definition of $u(F)$ and $u(H)$, this is only possible if $u.f = u(F)$ and $u(H) = u.h$. Therefore $f \epsilon F/u$, $h \epsilon H/u$.

If we had $f \geq h$ and $f \neq h$, then we would have $u.f > u.h$, which is not so because $u(F) \leq u(H)$. Therefore $f \leq h$.

Theorem 19. *If* $\mathfrak{J}$ *is a clear set of regular catalogues,* $F \epsilon \mathfrak{J}$, $H \epsilon \mathfrak{J}$, *and if for some* $\underline{u} \epsilon E$ *such that* $\underline{u} > 0$, *we have* $\underline{u}(F) < \underline{u}(H)$, *then for all* $u \epsilon E$ *such that* $u > 0$, *we likewise have* $u(F) < u(H)$.

Proof: It is clearly sufficient to prove the assertion for all $u \epsilon E$ such that $u > \underline{u}$. So let $\underline{u} < u$, $u \epsilon E$. According to Lemma 4 there are payoff vectors $f_0 \epsilon F/\underline{u}$, $h_0 \epsilon F/\underline{u}$ such that $f_0 \leq h_0$. Likewise there is an $f' \epsilon F/u$ and an $h' \epsilon H/u$ such that $f' \leq h'$ or $h' \leq f'$. We will now select a natural number $N \geq 1$ such that $N(\underline{u}(H) - \underline{u}(F)) > (u-\underline{u}).(f'-f_0)$. For $n = 0,\ldots,N$ we set $u = N^{-1}(N-n)\underline{u} + N^{-1}nu$. Then $u_0 = \underline{u}$, $u_N = u$ and for $n < N$, $(u_{n+1} - u_n) = N^{-1}(u - \underline{u})$. For $n = 0,\ldots,N$, according to Lemma 4 there are elements $f_n \epsilon F/u_n$ and $h_n \epsilon H/u_n$ such that $f_n \leq h_n$ or $h_n \leq f_n$; choose $f_N = f'$, $h_N = h'$. We have $u_0(F) < u_0(H)$.

We now prove that if $u_n(F) < u_n(H)$ for $n = 0,\ldots,m-1$, $1 \leq m \leq N$, then $u_m(F) < u_m(H)$. By complete induction we will then obtain $u_N(F) < u_N(H)$, and therefore, as asserted, $u(F) < u(H)$.

Let $0 < m \leq N$, $u_n(F) < u_n(H)$ for $n = 0,\ldots,m-1$. Then $f_n \leq h_n$ for $n = 0,\ldots,m-1$. We set $d_n = (u_{n+1}(H) - u_{n+1}(F)) - (u_n(H) - u_n(F))$ for $n = 0,\ldots,N-1$. Then $u_m(H) - u_m(F) = u_0(H) - u_0(F) + \Sigma_{n=1}^{m-1} d_n$. Now $d_n = (u_{n+1} - u_n).(h_n - f_n) + u_{n+1}.(h_{n+1} - h_n) - u_{n+1}.(f_{n+1} - f_n)$. By induction hypothesis, $h_n - f_n \geq 0$ for $n = 0,\ldots,m-1$. Moreover, it was assumed that $u > \underline{u}$, therefore $u_{n+1} - u_n > 0$. Consequently it follows that $(u_{n+1} - u_n).(h_n - f_n) > 0$ for $n = 0,\ldots,m-1$. Because $h_{n+1} \epsilon H/u_{n+1}$ and $h_n \epsilon H$, we have $u_{n+1}.(h_{n+1} - h_n) \geq 0$ for $n = 0,\ldots,N-1$. Similarly, $u_{n+1}.(f_{n+1} - f_n) \geq 0$ and $u_n.(f_{n+1} - f_n) \leq 0$, and a fortiori $(u_{n+1} - u_n).(f_{n+1} - f_n) \geq u_{n+1}.(f_{n+1} - f_n) \geq 0$ for $n = 0,\ldots,N-1$. We arrive at the estimate

$$\begin{aligned} u_m(H) - u_m(F) &> u_0(H) - u_0(F) - \Sigma_{n=0}^{N-1} N^{-1}(u-\underline{u}).(f_{n+1} - f_n) \\ &= \underline{u}(H) - \underline{u}(F) - N^{-1}(u-\underline{u}).(f'-f_0). \end{aligned}$$

However, N was so defined that the right side is greater than zero; so $u_m(H) > u_m(F)$. Thus the general inductive step has been proved. Hence $u(H) > u(F)$, and the proof is complete.

<u>Corollary 1. If</u> $\mathfrak{J}$ <u>is a clear set of regular catalogues,</u> $F\epsilon\mathfrak{J}$, $H\epsilon\mathfrak{J}$, <u>and if for some</u> $u\epsilon E$ <u>such that</u> $u > 0$ <u>we have</u> $u(F) = u(H)$, <u>then</u> $F = H$.

<u>Proof</u>: By Theorem 19, $u(F) = u(H)$ for all $u\epsilon E$ with with $u > 0$. For all $f\epsilon F$, $u.f \leq u(H)$ for all $u > 0$;

for all $h \in H$, $u.h \leq u(F)$ for all $u > 0$. Applying Lemma 2 twice, we obtain $F \subset H$ and $H \subset F$, therefore $F = H$.

Corollary 2. Every clear set of regular catalogues is completely ordered by inclusion.

Proof: If F and H are two catalogues of this set, and $u > 0$ is chosen arbitrarily, then $u(F) \leq u(H)$ or $u(H) \leq u(F)$. Theorem 19 in combination with Lemma 2 then yields $F \subset H$ or $H \subset F$.

Theorem 20. Every clear set of regular catalogues is r.o.

Proof: Let $\mathfrak{J}$ be an arbitrary clear set of regular catalogues. Then every nonempty finite subset of $\mathfrak{J}$ is also clear. Now let $\mathfrak{J}$ be a nonempty finite set of regular catalogues, F_1 = convex hull of $\bigcup_{F \in \mathfrak{J}} F$, $F_0 = \bigcap_{F \in \mathfrak{J}} F$ and $F_p = (1-p)F_0 + pF_1$ for $0 \leq p \leq 1$. $\mathfrak{J}$ is finite and, according to Corollary 2 of Theorem 19, ordered by inclusion. Therefore the intersection F_0 and the union of the catalogues in $\mathfrak{J}$ are themselves in $\mathfrak{J}$. The union is therefore convex, so F_1 is contained in $\mathfrak{J}$. Thus F_p is contained in $CH(\mathfrak{J})$ for all p, $0 \leq p \leq 1$. $CH(\mathfrak{J})$ is a clear set of regular catalogues (Theorems 11 and 16). Now let $F \in \mathfrak{J}$; then $F \in CH(\mathfrak{J})$. Choose $u \in E$ with $u > 0$. Because $F_0 \subset F \subset F_1$, $u(F_0) \leq u(F) \leq u(F_1)$. Therefore there is a p with $0 \leq p \leq 1$ such that $u(F) = (1-p)u(F_0) + pu(F_1)$.

Now $u(F_p) = (1-p)u(F_0) + pu(F) = u(F)$. F and F_p both belong to the clear set of regular catalogues $CH(\mathfrak{J})$; so by Corollary 1 of Theorem 19, they are equal, i.e., $F = F_p$. This is condition (r.o. 1).

Condition (r.o. 2) follows from Theorem 18 for $0 < p < 1$; for if $f_0 \epsilon F_0$ and $f_1 \epsilon F_1$ are such that $f = (1-p)f_0 + pf_1$ and $f_0 \geq f \geq f_1$, then because $F_0 \subset F_1$ we have $f_0 \epsilon F_1$; and by property (K3) of the catalogue F_0, $f_1 \epsilon F_0$. When $p = 0$ or 1, condition (r.o. 2) is easily verified.

If $\mathfrak{J}$ is an arbitrary clear set of regular catalogues, then according to what was just shown, every nonempty finite subset of $\mathfrak{J}$ is r.o. Then by Definition 19, $\mathfrak{J}$ is r.o.

Theorem 21. For sets $\mathfrak{J}$ of regular catalogues, the properties "$\mathfrak{J}$ is clear", "$\mathfrak{J}$ is r.o.", and "$\mathfrak{J}$ is classical" are equivalent.

Proof: If $\mathfrak{J}$ is a set of regular catalogues, then the following holds: If $\mathfrak{J}$ is clear, then $\mathfrak{J}$ is r.o. (Theorem 20). If $\mathfrak{J}$ is r.o., then $\mathfrak{J}$ is classical (Theorem 13). If $\mathfrak{J}$ is classical, then $\mathfrak{J}$ is clear (Theorem 8).

It can be shown that the set of Bernoullian catalogues is r.o., and therefore classical.

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